

possible rational capacities is given by $\{1/4, 1/2, 3/4\}$. A rate $3/4$ dc- and Nyquist-free code demonstrates the ability to code at the highest rational capacity available.

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Entropy and Power Spectrum of Asymmetrically DC-Constrained Binary Sequences

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Abstract—The eigenstructure of bidiagonal Hessenberg-Toeplitz matrices is determined. These matrices occur as skeleton matrices of finite-state machines generating certain asymmetrically dc-constrained binary sequences that can be used for simulating pilot tracking tones in digital magnetic recording. The eigenstructure is used to calculate the Shannon upper bound to the entropy of the finite state machine as well as the power spectrum of the maxentropic process generated by it.

Index Terms—Hessenberg-Toeplitz matrix, dc-constrained sequences, magnetic recording, entropy, power spectrum.

I. INTRODUCTION

We consider in this correspondence $(M+1) \times (M+1)$ bidiagonal Hessenberg-Toeplitz matrices

$$A = (A_{ij})_{i,j=0,\dots,M}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & \vdots \\ 1 & & & & \\ \vdots & & & & \\ 0 & & & & 1 & 0 \\ \vdots & & & & & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad (1.1)$$

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where $p = 1, 2, \dots$. These matrices arise as follows. Suppose we have a binary sequence $s_k = \pm 1$, a rational number $\alpha = q/p \in [0, 1]$ with $\text{GCD}(q, p) = 1$, and a $b > 0$ such that the constraint

$$t_n := \sum_{k=1}^n s_k - n\alpha \in [-b, b], \quad n \in \mathbb{Z} \quad (1.2)$$

is satisfied. We associate with t_n the finite-state machine with states l/p , $|l| \leq pb$. From state $t_n = l/p$ state $t_{n+1} = (l \pm p - q)/p$ can be reached, provided that $|l \pm p - q| \leq pb$, according as $s_{n+1} = \pm 1$. This gives rise to the skeleton matrix

$$\begin{array}{c} \xrightarrow{p-q} \\ \left[\begin{array}{cccccccc} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & & & & & & \\ 1 & & & & & & \\ 0 & \dots & & & & & 0 \\ \vdots & & & & & & \vdots \\ 0 & \dots & & 0 & 1 & 0 & \dots & 0 \end{array} \right] \\ \xleftarrow{M+1} \end{array} \quad (1.3)$$

where $M = 2 \lfloor pb \rfloor$, of which (1.1) is the special case with $q = p - 1$.

From the largest eigenvalue λ_0 of A the maximum Shannon entropy of the finite-state machine is found as $\log_2 \lambda_0$, and the transition matrix B for which the machine yields the maximum entropy is given by

$$B = (B_{ij})_{i,j=0,\dots,M} = \lambda_0^{-1} \begin{pmatrix} p_j \\ p_i A_{ij} \end{pmatrix}_{i,j=0,\dots,M}, \quad (1.4)$$

where $p = [p_0, \dots, p_M]^T$ is the right eigenvector of A corresponding to λ_0 . It will turn out that the left eigenvector of B corresponding to the eigenvalue 1 is a multiple of

$$q = [p_0 p_M, p_1 p_{M-1}, \dots, p_M p_0]. \quad (1.5)$$

This vector q is the vector of stationary probabilities. Finally, the power spectrum of the process t_n generated by the finite state machine with transition matrix B is proportional to

$$S_t(\theta) = \sum_{k=-\infty}^{\infty} R(|k|) e^{-2\pi i k \theta}, \quad (1.6)$$

where

$$R(k) = u^T F B^k u, \quad k = 0, 1, \dots, \quad (1.7)$$

with $F = \text{diag}(q_0, q_1, \dots, q_M)$ and $u = 1/p[-m, -m+1, \dots, m-1, m]^T$ ($m = \lfloor pb \rfloor$). The power spectrum $S_s(\theta)$ of the process $s_n = t_n - t_{n-1} + \alpha$ (the original binary data) is then given (since $E t_n = 0$) by

$$S_s(\theta) = \alpha^2 \delta(\theta) + 4 S_t(\theta) \sin^2 \pi \theta. \quad (1.8)$$

For the basic results concerning finite-state machines and their spectra we refer to [1], [4].

The main results of this correspondence are Theorem 1 and Theorem 2. In Theorem 1 all information on the eigenstructure of A is collected. This theorem gives rise to relatively simple computational schemes for the largest eigenvalue λ_0 of A , the vectors p and q previously discussed and the power spectrum $S_t(\theta)$ in (1.6). For the calculation of the power spectrum $S_s(\theta)$ this computational scheme has been worked out in Section IV.

It turns out that $S_f(\theta)$ can be decomposed as (with $M+1=2pN+R$)

$$S_f(\theta) = S_d(\theta) + \sum_{l=1}^{N-1} S_l(\theta) + S_{\text{nil}}(\theta). \quad (1.9)$$

Here $S_d(\theta)$ is the discrete component with spectral lines at $\theta = 1/2p, 2/2p, \dots, (2p-1)/2p$ (no spectral line at dc) that are due to the eigenvalues of A of largest modulus (excluding the largest positive eigenvalue). Furthermore, the $S_l(\theta)$, $l=1, \dots, N-1$, and $S_{\text{nil}}(\theta)$ constitute the continuous component of $S_f(\theta)$ and are due to the $2p$ eigenvalues of A of l th largest modulus and the R -fold eigenvalue 0 of A , respectively. These results can be considered as generalizations of those obtained by Chien [2], Justesen [4, Section V, Example 7], and Kerpez [5], in the sense that we consider certain $\alpha \neq 0$ and nonintegral b in (1.2). Unfortunately, we do not see how our approach can be extended to the more general matrices in (1.3).

An alternative way to evaluate $S_f(\theta)$ would be by using the general result in [2]. For this one needs the cyclic structure of A . Although this structure can be determined, and is even rather simple in this case, the resulting formulas for $S_f(\theta)$ are still less explicit than the ones we obtain.

We conclude this introduction by briefly mentioning a possible application of our results to pilot tracking tones in digital magnetic recording. In digital magnetic recorders part of the information storage capacity is exploited for recording servo position information. This information is often recorded as a low-frequency tone, usually called pilot tone, see [6]. The principle of operation is as follows. The frequencies of the pilot tones on even and odd tracks are different while their amplitudes are equal. As the reading head moves off track in one direction, the observed amplitude of one pilot tone decreases while the other increases. This information can then be used to drive back the head to the middle of the track. Since we are dealing with binary data, the obvious technique of adding a sinusoidal waveform to the data cannot be applied. Instead, one can create the effect of a block wave by storing over intervals of large, fixed length alternately a surplus and a deficit of positive symbols. The frequency of the block wave can be varied by appropriate choice of the interval lengths.

Two relevant questions to be answered are the following. 1) How is storage capacity decreased by including servo position information of this type? 2) How should one choose the frequencies of the pilot tones so as to avoid interference of pilot tones and data? This correspondence addresses these questions under the assumptions that the storage of the surplus/deficit of positive symbols is in accordance with (1.2) and that the data can indeed be stored in accordance with the maxentropic process associated with the transition matrix B in (1.4). Here the effect on the data spectrum of switching from the surplus mode to the deficit mode (and vice versa) has been neglected. It turns out that the maxentropic process gives rise to spectral lines at integer multiples of $1/2p$. Hence, one should choose the pilot tone frequencies away from the multiples to avoid interference of pilot tones and the data.

II. EIGENSTRUCTURE OF A

In this section we determine the eigenstructure of the matrix A in (1.1). Throughout we write $M+1=2pN+R$, $m=2lp+r$, where $R, r=0, 1, \dots, 2p-1$ and $N, l=0, 1, \dots$.

We define polynomials P_m , $m=0, 1, \dots$, according to the recursion

$$P_{m+1}(\lambda) = \lambda P_m(\lambda) - P_{m-2p+1}(\lambda), \quad m=0, 1, \dots \quad (2.1)$$

with the initialization $P_0 \equiv 1$, $P_k \equiv 0$, $k < 0$, also see [3]. Since

$$(A - \lambda I) \begin{bmatrix} P_0(\lambda) \\ \vdots \\ P_M(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -P_{M+1}(\lambda) \end{bmatrix}, \quad (2.2)$$

it is seen that $x \in \mathbb{C}^{M+1}$ is an eigenvector of A corresponding to $\lambda \in \mathbb{C}$ with nonvanishing 0th component if and only if x is a multiple of $[P_0(\lambda), \dots, P_M(\lambda)]^T$ with λ one of the zeros of P_{M+1} .

Proposition 1: Define for $m=2lp+r$

$$f_{l,r}(z) = \sum_{k=0}^l \binom{(2p-1)k+l+r}{l-k} (-1)^{l-k} z^k. \quad (2.3)$$

Then

$$P_m(\lambda) = \lambda^r f_{l,r}(\lambda^{2p}). \quad (2.4)$$

Proof: We have for the P_m 's the generating function

$$F(z, \lambda) = (1 - \lambda z + z^{2p})^{-1} = \sum_{m=0}^{\infty} P_m(\lambda) z^m, \quad (2.5)$$

which follows easily from (2.1). Then (2.4) follows on equating the coefficients of z^m in (2.5). \square

Proposition 2: All $f_{l,r}$ have l distinct positive zeros

$$\sigma_{0,l,r} > \sigma_{1,l,r} > \dots > \sigma_{l-1,l,r}.$$

Moreover,

$$\begin{aligned} \sigma_{0,l,r+1} > \sigma_{0,l,r} > \sigma_{0,l-1,r+1} > \sigma_{1,l,r+1} > \sigma_{1,l,r} > \dots > \\ > \sigma_{l-2,l-1,r+1} > \sigma_{l-1,l,r+1} > \sigma_{l-1,l,r} > 0 \end{aligned} \quad (2.6)$$

when $r=0, 1, \dots, 2p-2$, and

$$\begin{aligned} \sigma_{0,l+1,0} > \sigma_{0,l,2p-1} > \sigma_{0,l,0} > \sigma_{1,l+1,0} > \sigma_{1,l,2p-1} > \dots > \\ > \sigma_{l-1,l+1,0} > \sigma_{l-1,l,2p-1} > \sigma_{l-1,l,0} > \sigma_{l,l+1,0} > 0. \end{aligned} \quad (2.7)$$

Proof: The recursion (2.1) for P_m results in the recursion

$$f_{l,r+1}(z) = f_{l,r}(z) - f_{l-1,r+1}(z), \quad l=0, 1, \dots; r=0, 1, \dots, 2p-2,$$

$$f_{l+1,0}(z) = z f_{l,2p-1}(z) - f_{l,0}(z), \quad l=0, 1, \dots, \quad (2.8)$$

for $f_{l,r}$ according to (2.4). Note that $f_{l,r}(\infty) > 0$ for all $l, r \geq 0$.

The statements of the proposition and inequalities (2.6) and (2.7) follow from induction with respect to $m=2lp+r$. \square

Proposition 2 implies that the zeros of $f_{l,r}$ have an interlacing property and that $\sigma_{0,l,r}$ strictly increases in $m=2lp+r$.

Proposition 3: The nonzero eigenvalues of A are

$$\begin{aligned} \lambda_{2lp+r} &= \sigma_l^{1/2p} e^{2\pi i r / 2p}, \quad l=0, 1, \dots, N-1, \\ r &= 0, 1, \dots, 2p-1, \end{aligned} \quad (2.9)$$

and

$$x_{2lp+r} = (P_m(\sigma_l^{1/2p}) e^{2\pi i r m / 2p})_{m=0,1,\dots,M} \quad (2.10)$$

is the eigenvector of A corresponding to λ_{2lp+r} . Here, $\sigma_k = \sigma_{k,N,R}$ is the k th largest zero of $f_{N,R}$.

Proof: This follows easily from the preceding results. \square

Regarding the eigenvalue 0 of A we have the following result.

Proposition 4: Suppose $R > 0$ and put

$$y_r = [P_0^{(r)}(0), \dots, P_M^{(r)}(0)]^T, \quad r = 0, 1, \dots, R-1. \quad (2.11)$$

Then

$$Ay_0 = \mathbf{0}, \quad Ay_r = ry_{r-1}, \quad r = 1, \dots, R-1. \quad (2.12)$$

Proof: Let $r = 0, 1, \dots, R-1$, differentiate (2.2) r times and set $\lambda = 0$. Since P_{M+1} has an R th-order zero at 0 we obtain (2.12). \square

Note: We have

$$P_{2lp+s}^{(r)}(0) = \begin{cases} 0, & s \neq r, \\ r! f_{l,r}(0) = (-1)^l \frac{(l+r)!}{l!}, & s = r, \end{cases} \quad (2.13)$$

which shows e.g., that the y_r of Proposition 4 are orthogonal.

Proposition 5: Denote

$$X = [x_0 | \dots | x_{2Np-1}], \quad Y = [y_0 | \dots | y_{R-1}] \quad (2.14)$$

with x_m and y_r given in Propositions 3 and 4, respectively, and let Π be the $(M+1) \times (M+1)$ matrix defined by $(\Pi z)_m = z_{M-m}$ for

$$z = [z_0, z_1, \dots, z_M]^T \in \mathbb{C}^{M+1}.$$

Then we have $X^T \Pi Y = 0$, and

$$\Gamma := X^T \Pi X = \begin{bmatrix} \gamma_0 & 0 & \dots & 0 \\ 0 & \gamma_1 & & \vdots \\ \vdots & & & 0 \\ 0 & \dots & 0 & \gamma_{2pN-1} \end{bmatrix},$$

$$\Delta := Y^T \Pi Y = \begin{bmatrix} 0 & \dots & 0 & \alpha_{R-1} \\ \vdots & & & 0 \\ 0 & \alpha_1 & & \vdots \\ \alpha_0 & 0 & \dots & 0 \end{bmatrix}, \quad (2.15)$$

with

$$\gamma_m = P_{M+1}'(\lambda_m), \quad \alpha_r = (-1)^N (R-1-r)! r! \binom{R+N}{N}. \quad (2.16)$$

Proof: We have $\Pi A \Pi = A^T$. Hence, for $0 \leq m \leq 2Np-1$, $0 \leq r \leq R-1$

$$0 = x_m^T \Pi A^{r+1} y_r = \lambda_m^{r+1} x_m^T \Pi y_r. \quad (2.17)$$

This implies that $X^T \Pi Y = 0$. In what follows C_{z^M} abbreviates "coefficient of z^M in." We have for $\lambda \neq \mu$ by (2.5)

$$\sum_{j=0}^M P_j(\lambda) P_{M-j}(\mu) = C_{z^M} \left[\frac{1}{(1-\lambda z + z^{2p})(1-\mu z + z^{2p})} \right]$$

$$= \frac{P_{M+1}(\lambda) - P_{M+1}(\mu)}{\lambda - \mu}. \quad (2.18)$$

When we take $\lambda = \lambda_m$, $\mu = \lambda_n$ with $m \neq n$, the left-hand side of (2.18) becomes $x_m^T \Pi x_n$, while the right-hand side vanishes since

$$P_{M+1}(\lambda_m) = P_{M+1}(\lambda_n) = 0.$$

And when we take $\lambda = \lambda_m$ and $\mu \rightarrow \lambda_m$, we obtain $x_m^T \Pi x_m = P_{M+1}'(\lambda_m)$. This proves the claims about $X^T \Pi X$. To prove the claims about $Y^T \Pi Y$ we can use (2.13). Alternatively, we have as

in (2.18)

$$\sum_{j=0}^M P_j^{(r)}(\lambda) P_{M-j}^{(s)}(\lambda) = r! s! C_{z^M} \left[\frac{z^{r+s}}{(1-\lambda z + z^{2p})^{r+s+2}} \right]. \quad (2.19)$$

Setting $\lambda = 0$ and recalling that $M = 2pN + R - 1$ we easily get our result.

The next theorem summarizes the results of this section.

Theorem 1: Let $S = [X|Y]$ with X, Y as in (2.14). Then

$$S^{-1} A S = \begin{bmatrix} \Lambda & 0 \\ 0 & J \end{bmatrix} =: \Lambda_0, \quad (2.20)$$

where

$$\Lambda = \begin{bmatrix} \lambda_0 & 0 & \dots & 0 \\ 0 & \lambda_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_{2pN-1} \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & & \vdots \\ \vdots & & & \ddots & 0 \\ & & & & R-1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (2.21)$$

with λ_m given in Proposition 3. Moreover,

$$S^{-1} = \Gamma_0^{-1} S^T \Pi, \quad (2.22)$$

where

$$\Gamma_0 = \begin{bmatrix} \Gamma & 0 \\ 0 & \Delta \end{bmatrix}, \quad (2.23)$$

with Γ , Δ , and Π defined in Proposition 5.

We conclude this section by noting that finding the eigenstructure of A amounts to finding the N positive roots of the N th degree polynomial $f_{N,R}$ and plugging in these roots into the polynomials $f_{l,r}$ for $m = 2lp + r = 0, 1, \dots, M$. Moreover, the inversion of S , which is needed for efficient calculation of A^k , $k = 0, 1, \dots$, as required in Section IV, presents no difficulty by formula (2.22).

III. FINITE STATE MACHINE WITH MAXIMUM ENTROPY

In this section we make some comments on the entropy and vector of stationary probabilities of the maxentropic process associated with the skeleton matrix A in (1.1). The entropy of this process is $\log_2 \lambda_0^{(M)}$ (where we have explicitly indicated the dependence of the largest eigenvalue of A on M), and it has been shown in [3] that $\log_2 \lambda_0^{(M)}$ increases as a function of M to the limit¹

$$\log_2 \lambda_0^{(\infty)} = -\frac{2p-1}{2p} \log_2 \frac{2p-1}{2p} - \frac{1}{2p} \log_2 \frac{1}{2p}. \quad (3.1)$$

We observe that the right-hand side of (3.1) is the entropy of a binary channel generating independent ± 1 's with probabilities $(p \pm (p-1))/2p$ and average $\alpha = (p-1)/p$. We have more

¹After completion of this work, the authors became aware of a theorem of M. Biernacki, quoted in P. Schmidt, F. Spitzer, *Math. Scand.*, vol. 8, Sect. 7, pp. 15-38, 1960, that shows that a corresponding limit formula holds for the more general matrices in (1.3).

precisely, according to [3], the third-order approximation

$$\lambda_0^{(\infty)} \left(1 - \frac{\pi^2(2p-1)}{2(M+2)^2} \right) \quad (3.2)$$

to $\lambda_0^{(M)}$, and it can also be shown from [3], (3.20), that, stated somewhat imprecisely, the asymptotic form of the eigenvector $\mathbf{p} = [p_0, p_1, \dots, p_M]^T$ corresponding to $\lambda_0^{(M)}$ is given by

$$\left((2p-1)^{m/2p} \sin \frac{\pi(m+1)}{M+2} \right)_{m=0, \dots, M}, \quad M \rightarrow \infty. \quad (3.3)$$

Observe that for the case $p=1$ the vector in (3.3) is exactly the eigenvector of A corresponding to the largest eigenvalue (see [2])

$$2 \cos \frac{\pi}{M+2} = 2 \left(1 - \frac{\pi^2}{2(M+2)^2} + \dots \right). \quad (3.4)$$

To further appreciate these asymptotic results, we have calculated for $p=2$, $M+1=9$ (so that $N=2$, $R=1$) the quantities $\lambda_0^{(M)}$, $\lambda_0^{(\infty)}$ and the approximate (3.2). We obtain (since $f_{2,1}(z) = z^2 - 6z + 3$)

$$\begin{cases} \lambda_0^{(M)} = (3 + \sqrt{6})^{1/4} = 1.527878993, \\ \lambda_0^{(\infty)} = \left(\frac{4}{3} \right)^{3/4} 4^{1/4} = 1.754765351, \\ \lambda_0^{(\infty)} \left(1 - \frac{3\pi^2}{200} \right) = 1.494982753. \end{cases} \quad (3.5)$$

We conclude this section by noting that the left eigenvector \mathbf{q} of B in (1.4) corresponding to eigenvalue 1 (vector of stationary probabilities) is given by (1.5). This is an easy consequence of the facts that $\Pi A \Pi = A^T$ and that

$$B = \lambda_0^{-1} D^{-1} A D, \quad D = \text{diag}(p_0, p_1, \dots, p_M). \quad (3.6)$$

Note that \mathbf{q} is midpoint symmetric, i.e., $\Pi \mathbf{q} = \mathbf{q}$, and that the limiting form of \mathbf{q} , when $M \rightarrow \infty$, as follows from (3.3), is independent of p , viz. a multiple of

$$\left(\sin^2 \frac{\pi(m+1)}{M+2} \right)_{m=0, \dots, M} \quad (3.7)$$

IV. POWER SPECTRUM OF MAXENTROPIC PROCESS

In this section we present a decomposition of the power spectrum $S_i(\theta)$ of the maxentropic t_n of (1.2) in accordance with the eigenvalue structure of A . The power spectra $S_s(\theta)$ of the actual binary data and $S_i(\theta)$ are related according to (1.8). In view of (1.6) and (1.7) we have

$$S_i(\theta) = \sum_{k=-\infty}^{\infty} R(|k|) e^{-2\pi i k \theta}, \quad R(k) = \mathbf{u}^T F B^k \mathbf{u}, \quad k=0, 1, \dots, \quad (4.1)$$

where $F = \text{diag}(q_0, q_1, \dots, q_M)$ and $\mathbf{u} = p^{-1}[-m, -m+1, \dots, m-1, m]^T$ with $m = \lfloor pb \rfloor$. We have in the present case

$$M = 2 \lfloor pb \rfloor, \quad N = \left\lfloor b + \frac{1}{p} \right\rfloor, \quad R = 2 \lfloor pb \rfloor - 2p \left\lfloor b + \frac{1}{p} \right\rfloor + 1. \quad (4.2)$$

Lemma 1: Let (see Theorem 1 and (3.6))

$$S^T \Pi D \mathbf{u} =: \mathbf{v} = \begin{bmatrix} v_\sigma \\ v_{\text{nil}} \end{bmatrix} \quad (4.3)$$

with

$$\mathbf{v}_\sigma = [v_{\sigma,0}, v_{\sigma,1}, \dots, v_{\sigma,2pN-1}]^T, \quad \mathbf{v}_{\text{nil}} = [v_{\text{nil},0}, \dots, v_{\text{nil},R-1}]^T.$$

Then

$$R(k) = R_\sigma(k) + R_{\text{nil}}(k), \quad k=0, 1, \dots, \quad (4.4)$$

with

$$R_\sigma(k) = \sum_{l=0}^{N-1} \rho_l^k \sum_{r=0}^{2p-1} c_{l,r} e^{2\pi i r k / 2p}, \quad (4.5)$$

$$R_{\text{nil}}(k) = c_k. \quad (4.6)$$

Here

$$\rho_l = \frac{\lambda_{2lp}}{\lambda_0}, \quad c_{l,r} = -\gamma_{2lp+r}^{-1} v_{\sigma,2lp+r}^2, \quad (4.7)$$

and

$$c_k = -\lambda_0^{-k} \sum_{r=0}^{R-1-k} v_{\text{nil},r} v_{\text{nil},R-1-k-r} \frac{(r+1) \cdots (r+k)}{\alpha_{r+k}} \quad (4.8)$$

with the λ 's, γ 's and α 's given by Theorem 1.

Proof: Since $\Pi \mathbf{u} = -\mathbf{u}$ it follows from Theorem 1, (1.5), and (3.6) that

$$R(k) = \frac{-1}{\lambda_0^k} \mathbf{v}^T \Lambda_0^k \Gamma_0^{-1} \mathbf{v}. \quad (4.9)$$

The formulas (4.5–4.8) now follow from the results of Section II. \square

Lemma 1 shows that R_σ contains a periodic component (the term with $l=0$) and $N-1$ rapidly decaying components corresponding to the terms with $l=1, \dots, N-1$. Due to the special form of \mathbf{u} the numbers $c_{l,r}$ can be expressed (pretty much as the γ_m 's in (2.16)) in terms of (derivatives of) the P_m 's at special points. In particular, it can be shown that

$$c_{l,r}^* = c_{l,2p-r}; \quad c_{l,0}, c_{l,p} \in \mathbb{R}; \quad c_{0,0} = 0 \leq c_{0,r}. \quad (4.10)$$

As a consequence, certain terms in (4.5) can be combined, and we obtain the following result for the spectrum $S_i(\theta)$.

Theorem 2: We have

$$S_i(\theta) = S_d(\theta) + \sum_{l=1}^{N-1} S_l(\theta) + S_{\text{nil}}(\theta). \quad (4.11)$$

In this decomposition we have that

$$S_d(\theta) = \sum_{r=1}^{2p-1} c_{0,r} \delta \left(\theta - \frac{r}{2p} \right) \quad (4.12)$$

is the discrete component in $S_i(\theta)$, and that

$$S_l(\theta) = c_{l,0} P(\theta) + c_{l,p} P_l \left(\theta - \frac{1}{2} \right) + \sum_{r=1}^{p-1} \text{Re} \left[c_{l,r}^* T_l \left(\theta - \frac{r}{2p} \right) + c_{l,r} T_l \left(\theta + \frac{r}{2p} \right) \right], \quad l=1, \dots, N-1, \quad (4.13)$$

with

$$T_l(\theta) = \frac{1 - \rho_l^2 + 2i \rho_l \sin 2\pi \theta}{1 - 2\rho_l \cos 2\pi \theta + \rho_l^2}, \quad P_l(\theta) = \text{Re } T_l(\theta), \quad (4.14)$$

together with

$$S_{\text{nil}}(\theta) = c_0 + 2 \sum_{k=1}^{R-1} c_k \cos 2\pi k \theta \quad (4.15)$$

constitute the continuous component of $S_f(\theta)$. Here $\rho_l, c_{l,r}, c_k$ are given in Lemma 1.

Proof: This follows from Lemma 1 and (4.10). Furthermore, it is used that when $0 \leq \rho < 1$, $\theta, \varphi \in \mathbb{R}$, and

$$U_\rho(\varphi, \theta) = \sum_{k=-\infty}^{\infty} \rho^{|k|} e^{2\pi i k |\varphi - 2\pi i k \theta|}, \quad (4.16)$$

then for $A \in \mathbb{C}$

$$AU_\rho(\varphi, \theta) + A^*U_\rho(-\varphi, \theta) = \operatorname{Re} \left[A^* \frac{1 - \rho^2 + 2i\rho \sin 2\pi(\theta - \varphi)}{1 - 2\rho \cos 2\pi(\theta - \varphi) + \rho^2} + A \frac{1 - \rho^2 + 2i\rho \sin 2\pi(\theta + \varphi)}{1 - 2\rho \cos 2\pi(\theta + \varphi) + \rho^2} \right]. \quad (4.17)$$

And also it is used that

$$\sum_{k=-\infty}^{\infty} e^{2\pi i k |\varphi - 2\pi i k \theta|} = \frac{1}{2} \sum_{k=-\infty}^{\infty} [\delta(\theta - \varphi - k) + \delta(\theta + \varphi - k)] \quad (4.18)$$

with δ Dirac's delta function. Here convergence of both series is to be taken in the sense of generalized functions. \square

Note: The discrete component at 0 vanishes since $c_{0,0} = 0$.

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Spectral Lines of Codes Given as Functions of Finite Markov Chains

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Abstract—Spectral lines of signals that are given by functions of finite Markov chains are investigated. A problem of characterizing encoders is considered such that the messages emitted from these encoders have some amount of information about clock, independent of the source statistics. Necessary and sufficient conditions are established for the encoded message to have a spectral line of a given amplitude for every

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transition probability matrix of the underlying Markov chain. Several other related results about spectral lines are given. A biased coboundary condition is defined at a frequency f . It is also shown that this condition is necessary and sufficient for the encoded message to have a spectral density null at f .

Index Terms—Spectrum, spectral line, biased coboundary condition, finite state transition diagram.

I. INTRODUCTION

Some digital transmission systems require codes so that the frequency spectrum of the encoded message has a desired shape. For example, some digital magnetic recording systems use a code by which the source sequence is encoded into the message having a spectral null at dc.

In this correspondence we think of a directed graph with labeled states as a model of an encoder and a sequence of labels (complex numbers) along a path in the directed graph as an encoded sequence generated by the encoder. We consider the encoded message to be the function of the Markov chain given by a transition probability matrix compatible with the graph whose values are the sequences of labels along the infinite paths of the graph. It has been pointed out that the running digital sum at dc (denoted RDS_0) of the encoded sequence plays an important role in constructing and analyzing encoders whose encoded message has a small amount of frequency content at low frequencies, where for an encoded sequence $a = a_0 a_1 \cdots a_L$, RDS_0 of a is defined by

$$\text{RDS}_0(a) = \sum_{m=0}^L a_m.$$

Yasuda and Inose [16], [17] proved that for an encoder the following three conditions are equivalent: 1) the encoder satisfies a "finite RDS_0 condition" (i.e., for every encoded sequence from the encoder, RDS_0 takes its value in a finite range); 2) the encoder satisfies a "loop-sum-zero condition" (i.e., for every encoded sequence generated by a cycle of the encoder, its RDS_0 value is 0); 3) the encoded message has a spectral null at dc (i.e., the encoded message has zero mean and the power spectral density vanishes at dc). This was rediscovered by [7], [13]. Let k be a nonnegative integer and n a positive integer with $\gcd(k, n) = 1$. Let f_s be the symbol frequency. Yoshida and Yajima [19] defined the running digital sum at $f = kf_s/n$ (denoted RDS_f) of an encoded sequence $a = a_0 a_1 \cdots a_L$ to be

$$\text{RDS}_f(a) = \sum_{m=0}^L a_m \exp\left(-j2\pi \frac{k}{n} m\right), \quad (1.1)$$

where $j = \sqrt{-1}$. (They call $\text{RDS}_f(a)$ a weighted digital sum variation of a .) As an extension of the previous result, Yoshida and Yajima [18], and Marcus and Siegel [10] proved that the following three conditions are equivalent: 1) the encoder satisfies a "finite RDS_f condition"; 2) for every encoded sequence generated by a cycle of the encoder of length a multiple of n , its RDS_f value is 0; 3) the encoded message has a spectral null at f (i.e., the encoded message has a zero spectral line and the power spectral density vanishes at f). Moreover, Marcus and Siegel showed that a "coboundary condition at f " is equivalent to these three conditions, and that the coboundary condition is useful in constructing encoders for a spectral null at f (canoni-