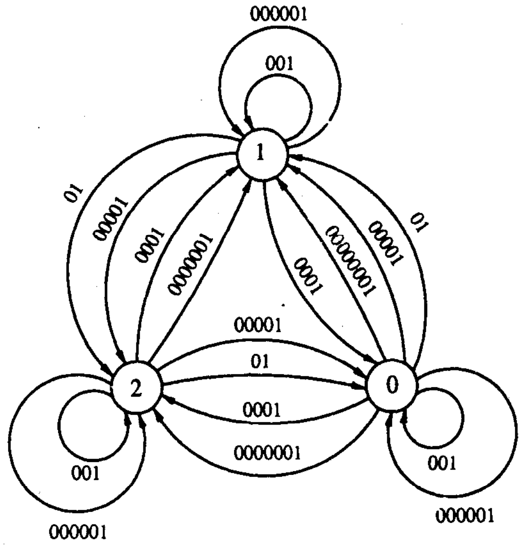

Fig. 10. Runlength constraint graph.

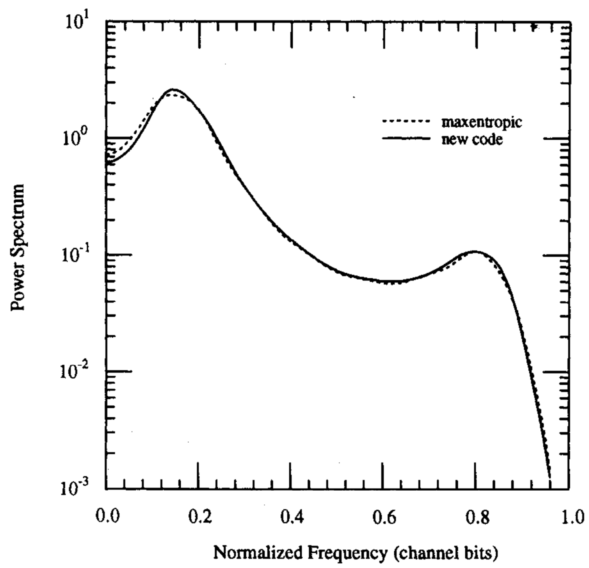

Fig. 11. Continuous time power spectra of maxentropic (1,7) constraint and new (1,7) code.

and the IBM code independently [2], [10]. The runlength graphs were identical and the result is shown in Fig. 10. The power spectrum resulting from the graph is shown in Fig. 11 along with the power spectrum for the maxentropic case. It turns out that the constraint graph in Fig. 5 corresponds exactly to one of the unused components of the cubed (1,7) constraint graph in [2].

VI. Summary

A new (1,7) encoder has been presented along with a corresponding decoder. The finite state machine that generates the code maps two user bits into three channel bits. The encoder requires 2 state variables in order to keep track of the internal state. The decoding algorithm is implemented by means of a sliding block decoder that requires no more than 8 channel bits to produce an estimate of the current 2 user bits. The error propagation due to a single-channel bit error is limited to 5 bits. The sequences that can be generated by this encoder are identical to those generated by the industry standard, the IBM (1,7) code. The statistical properties are identical as well. As far as the channel is concerned, the performance of the new code should be equivalent to that of the IBM code. If any difference exists, it would have to be caused by the mapping between user bits and channel bits. However, there is no indication that any such performance difference exists.

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Performance of Efficient Balanced Codes

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Abstract—**Recently, Knuth presented coding schemes in which each codeword contains equally many "zeros" and "ones." The construction has the virtue that the codes can be efficiently encoded and decoded. In this correspondence, we address the problem of appraising the spectral performance of codes based on Knuth's algorithm.**

Index Terms—**dc-balanced codes, Knuth codes, sum variance.**

I. Introduction

Binary sequences with spectral nulls at zero frequency have found widespread application in optical and magnetic recording systems. These *dc-balanced codes*, as they are often called, have a long history and their application is certainly not confined to recording practice. Since the early days of digital communication over cable, dc-balanced codes have been employed to

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counter the effects of low-frequency cut-off due to coupling components, isolating transformers, etc. In optical recording, as explained in [3], dc-balanced codes are employed to circumvent or reduce interaction between the data written on the disc and the servo systems that follow the track. A present-day application of dc-free codes [4] is the digital audio tape recorder which uses an 8b10b code to circumvent the effects of crosstalk from adjacent tracks, and to minimize over-write noise.

Practical coding schemes devised to achieve suppression of low-frequency components are mostly constituted by block codes. The (bipolar) source digits are grouped in source words of m digits; the source words are translated using a conversion table known as a codebook into blocks of n digits. The essential principle of operation of a channel encoder that translates arbitrary source data into a dc-free channel sequence is remarkably simple. The approaches that have actually been used for dc-balanced code design are basically three in number: zero-disparity code, low-disparity code, and polarity bit code.

The *disparity* of a codeword [6] is defined as the sum of its digits; thus the codewords $-1,-1,-1,1,1,-1$ and $1,-1,-1,1,1,1$ have disparity -2 and $+2$, respectively. Of special interest are zero-disparity codewords. The obvious method for the construction of dc-balanced codes is to only employ zero-disparity codewords.

A logical step, then, is to extend this mechanism to the *low-disparity* code, where the translations are not one-to-one. The source words operate with two alternative translations (or modes) that are of equal or opposite disparity; each of the two modes is interpreted by the decoder in the same way. The zero-disparity words are uniquely allocated to the source words. Other codewords are allocated in pairs of opposite disparity. During transmission, the choice of a specific translation is made in such a way that the accumulated disparity, or the *running digital sum*, of the encoded sequence, after transmission of the new codeword, is as close to zero as possible. The running digital sum (RDS) is defined as the accumulated sum of the transmitted digits, counted from the start of the transmission. Both of the basic approaches to dc-balanced coding are due to Cattermole [5], [6], and Griffiths [7].

A third coding method (in fact a special case of low-disparity codes), known as the *polarity bit code*, was devised by Bowers [8] and Carter [9]. They proposed a slightly different construction of dc-balanced codes as being attractive because no look-up tables are required for encoding and decoding. In their method, each group of $(n-1)$ source symbols are supplemented by the symbol "one." The encoder has the option to transmit the resulting n-bit words without modification or to invert all symbols. Like in the low-disparity code, the choice of a specific translation is made in such a way that the accumulated disparity is as close to zero as possible. The last symbol of the codeword, called the polarity bit, is used by the decoder to identify whether the transmitted codeword has been inverted or not.

Quite recently [2], a new algorithm for generating zero-disparity codewords was presented by Knuth. The method is based on a simple correspondence between the set of all m-bit source words and the set of all $(m+p)$-bit balanced codewords. The translation is in fact achieved by selecting a bit position within the m-bit word which defines two segments, each having one half of the total block disparity. A zero-disparity block is now generated by the inversion of all the bits within one segment. The remaining p bits of the codeword contain a balanced encoding of the bit position that defines the two segments. For a precise description of this method, we refer to Section II.

The outline of this paper is as follows. In order to get some insight into the efficiency of Knuth's construction technique we shall evaluate the spectral properties of its code streams. Of course, the spectrum may be evaluated for any given code structure by resorting to numerical computation. The theory provided in [10] furnishes efficient procedures for the computation of the power spectral density function of block-coded signals produced by an encoder that can be modeled by a finite-state machine. However, the computational load of this procedure is enormous if $m \gg 1$. Fortunately, the structure of Knuth codes allows us to derive a simple expression for (an approximation to) the *sum variance* of these code. This quantity plays a key role in the spectral performance characterization of dc-balanced codes, as explained in Section III. We shall evaluate this expression in Section IV. In Section V, we compare the sum variance of Knuth codes with the sum variance of the polarity bit codes, for fixed redundancy.

II. Balancing of Codewords

Most schemes for generating dc-balanced sequences use look-up tables, and are therefore restricted to codewords of medium size. An alternative and easily implementable encoding technique for zero-disparity codewords that is capable of handling (very) large blocks was described by Knuth [2]. The method is based on the idea that there is a simple correspondence between the set of all m-bit binary source words and the set of all $(m+p)$-bit codewords. For the sake of convenience, we will assume in the sequel that both m and p are even. (Similar constructions as described here are possible if one or both of p,m are odd.) Then the translation can in fact be achieved by selecting a bit position within the m-bit word that defines two segments, each having one half of the total block disparity. A zero-disparity block is now generated by the inversion of all the bits within one segment. The position information which defines the two segments is encoded in the p bits by a balanced word.

Let $z_k(x)$ be the running digital sum of the first k, $k \le m$, bits of the binary source word $x=(x_1,\cdots,x_m)$, $x_i \in \{-1,1\}$, or

$$z_k(x)=\sum_{i=1}^{k} x_i, \tag{1}$$

and let $x^{[k]}$ be the word x with its last $m-k$ bits inverted. (Note that the quantity $z_m(x)$ is the *disparity* of x.) For example, if

$$x=(-1,1,1,1,-1,1,-1,1,1,-1),$$

then the disparity of x equals 2 and

$$x^{[4]}=(-1,1,1,1,1,-1,1,-1,-1,1).$$

If we let $\sigma_k(x)$ stand for the disparity $z_m(x^{[k]})$ of $x^{[k]}$, then the quantity $\sigma_k(x)$ is

$$\begin{aligned}\sigma_k(x)&=\sum_{i=1}^{k} x_i-\sum_{i=k+1}^{m} x_i\\&=-z_m(x)+2\sum_{i=1}^{k} x_i.\end{aligned} \tag{2}$$

As an immediate consequence, $\sigma_0(x)=-z_m(x)$, (all symbols inverted) and $\sigma_m(x)=z_m(x)$ (no symbols inverted). If x is of even length m, then we may conclude that every word x can be associated with at least one k for which $\sigma_k(x)=0$, or $x^{[k]}$ is balanced. The value of the first such k is encoded in a balanced word of length p, p even. (If m and p are both odd, a similar construction is possible.) The maximum codeword length $m+p$ is governed by

$$m \leq \binom{p}{p/2}. \tag{3}$$

Some other modifications of the basic scheme are discussed in [2] and [11].

III. Spectrum and Sum Variance of Sequences

Let $\{x_i\}_{i \geq 0}$ denote a cyclo-stationary channel sequence. The power spectral density function of the sequence is given by [16]

$$H(\omega) = R(0) + 2\sum_{i=1}^{\infty} R(i)\cos i\omega, \qquad -\pi \leq \omega \leq \pi, \tag{4}$$

where $R(i) = E\{x_j x_{j+i}\}$, $i = 0, \pm 1, \pm 2, \cdots$ is the auto-correlation function of the sequence. In the sequel it is assumed that $\{x_i\}_{i \geq 0}$ is composed of cascaded codewords x of length n. Let $x^{(k)} = (x_1^{(k)}, \cdots, x_n^{(k)})$, $x_i^{(k)} \in \{-1, 1\}$, be the kth element of a set S of codewords. The Fourier transform $X^{(k)}(\omega)$ of the codeword $x^{(k)}$ is defined by [15]

$$X^{(k)}(\omega) = \sum_{i=1}^{n} x_i^{(k)} e^{-ji\omega}, \qquad -\pi \leq \omega \leq \pi, \tag{5}$$

where $j = \sqrt{-1}$. For all i, $1 \leq i \leq n$, let the number of codewords $x^{(k)} \in S$ with $x_i^{(k)} = 1$ be equal to the number of codewords with $x_i^{(k)} = -1$ (i.e., the sum of all codewords in S is the all-zero vector). If in addition, codewords are randomly chosen from S to form an infinite sequence, then it is rather straightforward to show, following [16] and [17], that the power spectral density function $H(\omega)$ of the concatenated sequence is

$$H(\omega) = \frac{1}{n|S|} \sum_{x^{(k)} \in S} \left| X^{(k)}(\omega) \right|^2. \tag{6}$$

Note that, due to our assumptions, $|S|$ has to be even. We will assume that the codewords are equiprobable.

From now on, we shall assume that the set of codewords S forms a *dc-balanced code*, in other words, the disparity of all the members of S is zero. The width of the spectral notch is a relevant quantity since it specifies the frequency region of suppressed components around the spectral null. Let $H(\omega)$ denote the power spectral density function of the sequences formed by cascading the codewords. Then the width of the spectral notch is defined by a quantity called the *cut-off* frequency. The cut-off frequency, denoted by ω_0, is defined by [13], [14]

$$H(\omega_0) = \frac{1}{2}. \tag{7}$$

Another quantity used in the performance evaluation of codes with a null at the zero frequency is the running digital sum [12]. The running digital sum Z_j is defined by

$$Z_j = \sum_{i=0}^{j} x_i. \tag{8}$$

Specifically, the variance of the running digital sum, in short sum variance, plays a key role in the spectral performance characterization of dc-balanced codes. The sum variance of the encoded sequence is defined as

$$s^2 = E\{Z_i^2\}, \tag{9}$$

where $E\{\cdot\}$ denotes the expectation operator. It was found by Justesen [13] that for dc-free sequences the product of cut-off frequency ω_0 and the sum variance of the sequence is approximately $1/2$, or

$$2\omega_0 s^2 \approx 1. \tag{10}$$

It has been found that this relationship between the sum variance and the actual cut-off frequency is accurate within a few percent [12]. On the other hand, the sum variance is relevant in its own right as it gives the variance of the intersymbol interference when the channel is ac-coupled.

As in the case of $H(\omega)$, it can be shown that, under similar conditions, the sum variance s^2 of the concatenated sequence is [12]

$$s^2 = \frac{1}{n|S|} \sum_{x^{(k)} \in S} \sum_{j=1}^{n} \left[z_j(x^{(k)}) \right]^2, \tag{11}$$

where $z_j(x) := \sum_{i=1}^{j} x_i$ and $|S|$ denotes the *cardinality* of S. (Observe that $|S| = 2^m$.)

In principle, the above equations can be invoked to compute the spectrum and sum variance of sequences generated by Knuth's method. In that case, S consists of the collection of $(m+p)$-bit codewords obtained from the set of all m-bit source words as described in Section II. Naturally, the above operation of enumerating all codewords is, for large codeword sets, a considerable computational load. (Recall that $|S| = 2^m$.) The computational load can be alleviated by making some approximations. Ignoring the contribution from the p-bit vector (note that Knuth's algorithm is especially designed for large codewords, i.e., $p \ll m$), the sum variance can be approximated as

$$s^2 \approx \frac{1}{n|S|} \sum_{y \in S} \sum_{j=1}^{m} \left[z_j(y) \right]^2. \tag{12}$$

This approximation is justified mathematically by the facts that

1) $z_m(y) = 0$, for $y \in S$, and
2) $1/n \sum_{j=1}^{p} [z_j(u)]^2 = O(p^3/m) = o(m), (m \to \infty)$, for a balanced p-bit word u.

In the next section we shall evaluate the right-hand side in (12) exactly.

IV. A Counting Problem

In this section we shall provide an exact evaluation of the right-hand side of the approximation for s^2 in (12). We first introduce some useful notation. Throughout this section, m denotes an even, positive integer. The collection of all binary words $x = (x_1, \cdots, x_m)$, $x_i \in \{-1, 1\}$, will be denoted by R_m. For $x \in R_m$, we define $z(x) = (z_0(x), \cdots, z_m(x))$ by

$$z_k(x) := \sum_{i=1}^{k} x_i, \tag{13}$$

$k = 0, \cdots, m$. (Note that $z_0(x) = 0$ by convention. Note also that $z_k(x) \equiv k \bmod 2$.) Put

$$S_m := \{ z(x) | x \in R_m \}.$$

For each integer k, $-m \leq k \leq m$, we let

$$R_m(k) := \{ x \in R_m | z_m(x) = k \}$$

and

$$S_m(k) := \{ z \in S_m | z_m = k \}.$$

Let $x \in R_m$. The quantity $z_k(x) - (1/2) z_m(x)$ starts at $-(1/2) z_m(x)$ for $k = 0$, ends at $(1/2) z_m(x)$ for $k = m$, and

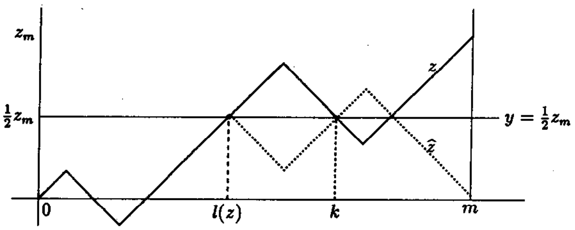

Fig. 1. Relation between z and $\hat{z}$.

increases by ± 1 if k increases by 1. Since m is even, we conclude that there exists a smallest integer $l = l(x)$, $0 \le l(x) \le m-1$, for which $z_l(x) = (1/2)z_m(x)$. Define $\hat{x}$ by

$$\hat{x} := (x_1, \cdots, x_l, -x_{l+1}, \cdots, -x_m), \tag{14}$$

with $l = l(x)$. Observe that in Knuth's method a word x is encoded as a word with initial part $\hat{x}$. Observe also that $\hat{x} \in R_m(0)$, in other words, $\hat{x}$ is balanced.

Next, let $z \in S_m$, with $z = z(x)$, say. (Note that x is uniquely determined by z.) By abuse of notation, we set $l(z) := l(x)$, and we let $\hat{z}$ be defined by $\hat{z} := z(\hat{x})$. In other words,

$$\hat{z}_k := \begin{cases} z_k, & \text{if } 0 \le k \le l(z), \\ z_m - z_k, & \text{if } l(z) \le k \le m. \end{cases} \tag{15}$$

Note that $\hat{z} \in S_m(0)$ by definition. Moreover, observe that $\hat{z}$ can be obtained from z by a reflection of the part $(z_l, \cdots, z_m)$ of z with respect to the line $y = (1/2)z_m$. (See Fig. 1.)

With the definitions, the approximation (12) can now be expressed as

$$s^2 \approx \frac{1}{n2^m}\lambda(m), \tag{16}$$

where the quantity $\lambda(m)$ is defined as

$$\lambda(m) := \sum_{z \in S_m} \sum_{j=1}^{m} \hat{z}_j^2. \tag{17}$$

Our aim is to prove the following result.

Theorem 1: $\lambda(m) = m(3m+2)2^{m-4}$.

The proof of Theorem 1 will depend on a number of lemmas and uses induction on m. (It is easily verified that Theorem 1 indeed holds for small values of m.) In the course of our computations, we will frequently make use of the following two results. Let η_{ij} and ρ_{ij}, $1 \le i, j \le m$, be defined by

$$\eta_{ij} := 2^{-m} \sum_{x \in R_m} x_i x_j, \qquad \rho_{ij} := \binom{m}{m/2}^{-1} \sum_{x \in R_m(0)} x_i x_j. \tag{18}$$

Lemma 1: We have

$$\eta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Proof: Evident. □

Lemma 2: We have

$$\rho_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -1/(m-1), & \text{otherwise.} \end{cases} \tag{20}$$

Proof: By symmetry, ρ_{ij} only depends on whether or not $i = j$ holds. Moreover, from the definition of $R_m(0)$ we find that $\sum_{j=1}^{m} \rho_{ij} = 0$. Since it is immediate that $\rho_{ii} = 1$, the result now follows. □

Lemmas 1 and 2 indicate that it is easy to compute a sum over S_m or $S_m(0)$ of polynomial functions of the z_j only (i.e., not involving the $\hat{z}_j$), simply by writing out the sum in terms of the x_j.

Our next result simplifies the expression (17) for $\lambda(m)$.

Lemma 3: We have

$$\lambda(m) = \sum_{z \in S_m} z_m \sum_{j=1}^{m} \hat{z}_j.$$

Proof: It will be sufficient to show that

$$\sum_{z \in S_m} \left(\hat{z}_j^2 - z_m \hat{z}_j\right) = 0 \tag{21}$$

holds, for all j, $1 \le j \le m$. To show this, we first observe that

$$\left(\hat{z}_j - \frac{1}{2}z_m\right)^2 = \left(z_j - \frac{1}{2}z_m\right)^2, \tag{22}$$

(see also Fig. 1), and thus

$$\hat{z}_j^2 - z_m \hat{z}_j = z_j^2 - z_m z_j. \tag{23}$$

So we are finished if we can show that

$$\sum_{z \in S_m} \left(z_j^2 - z_m z_j\right) = 0. \tag{24}$$

But that is easy, either by using Lemma 1, or by observing that the total contribution to the sum in (24) of $z(x)$ and $z(x')$ is 0, where $x' := (x_1, \cdots, x_j, -x_{j+1}, \cdots, -x_m)$. □

At this point, there are several ways to proceed. In view of Lemma 1, it would be sufficient to derive a closed-form expression for the quantity

$$\sum_{z \in S_m} z_m \sum_{j=1}^{m} (z_j - \hat{z}_j). \tag{25}$$

(This approach is, in a sense, the most logical way to proceed.) It is indeed possible to evaluate this quantity (using Andre's reflection principle, see for example [18]) in terms of a complicated double summation involving the product of certain binomial coefficients. Unfortunately, although we knew what the outcome should be we were unable to find a direct proof.

Here, we will follow another approach that we now describe. For each integer k, $0 \le k \le m$, we let

$$S_m^{(k)} := \left\{ z \in S_m \middle| z_k = \frac{1}{2}z_m \right\},$$

and for $z \in S_m^{(k)}$, we define $z^{(k)} = (z_0^{(k)}, \cdots, z_m^{(k)})$ by

$$z_j^{(k)} := \begin{cases} z_j, & \text{if } 0 \le j \le k, \\ z_m - z_j, & \text{if } k < j \le m. \end{cases} \tag{26}$$

Observe that $z^{(k)} \in S_m(0)$ for all $z \in S_m^{(k)}$, and $z^{(k)} = \hat{z}$ if $k = l(z)$. So we may write

$$\begin{aligned} \lambda(m) &= \sum_{z \in S_m} z_m \sum_{j=1}^{m} \hat{z}_j \\ &= \sum_{k=0}^{m} \sum_{z \in S_m^{(k)}} z_m \sum_{j=1}^{m} z_j^{(k)} - \sum_{k=0}^{m} \sum_{0 \le l < k} \sum_{\substack{x \in S_m^{(k)} \\ l(z) = l}} z_m \sum_{j=1}^{m} z_j^{(k)}. \end{aligned} \tag{27}$$

(To see this, note that each $z \in S_m$ contributes $z_m \sum_{j=1}^{m} z_j^{(k)}$ to the first summation in (27) for each k such that $z \in S_m^{(k)}$. By definition of $l(z)$, the smallest such k equals $l(z)$ and gives a contribution equal to $z_m\Sigma_{j=1}^{m}\hat{z}_j$, while the contributions for all $k>l$ occur also in the second summation in (27).) We will evaluate the two parts of (27) for $\lambda(m)$ separately.

Lemma 4:

$$\sum_{k=0}^{m}\sum_{z\in S_m^{(k)}} z_m\sum_{j=1}^{m} z_j^{(k)} = 2\sum_{u\in S_m(0)}\left(\sum_{j=1}^{m}u_j\right)^2 = \frac{1}{6}m^2(m+1)\binom{m}{m/2}.$$

Proof: For fixed k, the map $z\mapsto z^{(k)}$ defines a one-to-one correspondence between $S_m^{(k)}$ and $S_m(0)$. (Indeed, the inverse of this map is the map $u\mapsto(u_1,\cdots,u_k,2u_k-u_{k+1},\cdots,2u_k-u_{m-1},2u_k)$.) Since $z_m=2z_k=2z_k^{(k)}$ for all $z\in S_m^{(k)}$ by definition of $S_m^{(k)}$, the first equation follows.

To evaluate the second sum, write

$$\sum_{u\in S_m(0)}\left(\sum_{j=1}^{m}u_j\right)^2 = \sum_{x\in R_m(0)}\left(\sum_{j=1}^{m}\sum_{i=1}^{j}x_i\right)^2 \tag{28}$$

and use Lemma 2, together with the well-known series $\Sigma_{k=0}^{m}k=(1/2)m(m+1)$ and $\Sigma_{k=0}^{m}k^2=(1/6)m(m+1)(2m+1)$. We leave all further details to the reader. □

Lemma 5:

$$\sum_{k=0}^{m}\sum_{0\le l<k}\sum_{\substack{z\in S_m^{(k)}\\ l(z)=l}} z_m\sum_{j=1}^{m}z_j^{(k)} = \sum_{\substack{n=2\\ n\equiv 0 \bmod 2}}^{m-2}\binom{n}{n/2}\left\{\lambda(m-n)+\frac{1}{2}n(m-n)2^{m-n}\right\}.$$

Proof: Fix k and l, with $0\le l<k$, and put $n:=k-l$. Let $z\in S_m^{(k)}$, with $l(z)=l$. (Observe that this is possible only if n is even.) With each such z, we associate a pair of sequences $a(z)$ and $b(z)$, defined as follows.

$$a_j(z):=\begin{cases} z_j, & \text{if } 0\le j\le l,\\ z_{j+n}, & \text{if } l<j\le m-n,\end{cases} \tag{29}$$

$$b_j(z):=z_{j+l}-\frac{1}{2}z_m,\qquad 0\le j\le n. \tag{30}$$

Observe that $b(z)$ represents the part of z between indexes l and k, and since $z_k=z_l=(1/2)z_m$ by our assumptions on z, it follows that $b(z)\in S_n(0)$. Observe also that $a(z)$ can be obtained by removing from z the part of z between indexes $l+1$ and k, and thus $a(z)\in S_{m-n}$, $l(a(z))=l(z)=l$ and $a_{m-n}(z)=z_m$. (See again Fig. 1.) From these observations, and from the definition of $z^{(k)}$, we find that

$$z_m\sum_{j=1}^{m}z_j^{(k)} = a_{m-n}(z)\left\{\sum_{j=1}^{m-n}\widehat{a_j(z)} + \sum_{i=1}^{n}\left[b_i(z)+\frac{1}{2}a_{m-n}(z)\right]\right\}. \tag{31}$$

Conversely, each pair a and b for which $a\in S_{m-n}$, $l(a)=l$ and $b\in S_n(0)$ determines, through (29) and (30), a unique $z\in S_m^{(k)}$ with $l(z)=l$ such that $a(z)=a$ and $b(z)=b$. By this observation, we find from (31) that

$$\begin{aligned}\sum_{\substack{x\in S_m^{(k)}\\ l(z)=l}} z_m\sum_{j=1}^{m}z_j^{(k)} &= \sum_{\substack{a\in S_{m-n}\\ l(a)=l}}\sum_{b\in S_n(0)} a_{m-n}\left\{\sum_{j=1}^{m-n}\widehat{a_j}+\sum_{i=1}^{n}b_i+\frac{1}{2}na_{m-n}\right\}\\ &= \binom{n}{n/2}\sum_{\substack{a\in S_{m-n}\\ l(a)=l}}\left\{a_{m-n}\sum_{j=1}^{m-n}\widehat{a_j}+\frac{1}{2}na_{m-n}^2\right\},\end{aligned} \tag{32}$$

where the last equation follows from the trivial observation that

$$\sum_{b\in S_n(0)} b_i=0, \tag{33}$$

$1\le i\le n$. (Indeed, $b\in S_n(0)$ if and only if $-b\in S_n(0)$, and b and $-b$ together contribute 0 to the sum in (33).)

Finally, observe that

$$\sum_{a\in S_{m-n}} a_{m-n}\sum_{j=1}^{m-n}\widehat{a_j}=\lambda(m-n)$$

by definition of $\lambda(m-n)$, and

$$\sum_{a\in S_{m-n}} a_{m-n}^2=(m-n)2^{m-n}.$$

(The last statement can easily be proved using Lemma 1.) Then, since $z_m=0$ whenever $z\in S_m^{(m)}$, the contribution from $k=m$ to the sum in the lemma is 0, so the lemma now follows from (32) by summation over l and n. (Substitute $k=n-l$.) □

If we now combine Lemmas 4 and 5 with the expression for $\lambda(m)$ in (27), and if we write $m=2M$ and $n=2k$, we find the following result.

Corollary 1:

$$\sum_{k=0}^{M-1}\binom{2k}{k}\{\lambda(2M-2k)+2k(M-k)4^{M-k}\} = \frac{2}{3}M^2(2M+1)\binom{2M}{M}.$$

By the induction hypothesis of Theorem 1, we may assume that

$$\lambda(2M-2k)=(2M-2k)(6M-6k+2)4^{M-k-2}, \tag{34}$$

for $k=1,\cdots,M-1$. Therefore, in order to prove Theorem 1, it remains only to be shown that substitution of the relation (34), for *all* values $k=0,\cdots,M-1$, into the expression in Corollary 1 results in an identity. This is indeed the case that follows from the fact that

$$\sum_{k=0}^{M-1}(k)_e\binom{2k}{k}4^{M-k-1}=\frac{(M)_{e+1}}{2(2e+1)}\binom{2M}{M}, \tag{35}$$

($e=0,1,2,\cdots$), where $(k)_e:=k(k-1)\cdots(k-e+1)$ for $e\ge 1$ and $(k)_0:=1$. This easily follows from the well-known expansion

$$F(x):=(1-x)^{-1/2}=\sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{x}{4}\right)^k \tag{36}$$

by calculating $(1-x)^{-1}F^{(e)}(x)$. (The instances $e=0,1,2$ of (35), which we need here, can also be derived for example from [1, p. 613, no. 19].) Further details are left to the reader.

TABLE I
RATE AND SUM VARIANCE OF KNUTH CODES AND POLARITY BIT CODES VERSUS CODEWORD LENGTH $n = m + p$

p	m	$1-R$	s_K^2	n_p	s_p^2
6	20	0.2308	3.875	5	3
8	70	0.1026	13.250	10	6.33
10	252	0.0382	47.375	27	17.67
12	924	0.0128	173.375	78	51.67
14	3432	0.0041	643.625	247	164.33
16	12870	0.0012	2413.250	806	537
18	48620	0.0004	9116.375	2703	1801.67

V. PERFORMANCE APPRAISAL

We are now in a position to compare the performance of Knuth codes with that of other schemes. Specifically, we will compare the performance of Knuth codes with that of the polarity bit scheme. Both schemes, Knuth codes and polarity bit codes, have the property that they can be encoded and decoded without look-up tables. (To be precise, in the case of Knuth codes the encoding of the value of l into a p-bit balanced word can be effectuated either by using a *small* look-up table or algorithmically by enumerative encoding [19].)

The rate of Knuth codes as a function of p is at most

$$R = \frac{m}{m+p}, \tag{37}$$

where $m = \binom{p}{p/2}$. Let n_p denote the length of a codeword in the polarity bit code. The rate of the polarity bit code is

$$R = 1 - \frac{1}{n_p}. \tag{38}$$

The sum variance of a bit stream generated by the polarity bit format is [12]

$$s_p^2 = \frac{2n_p - 1}{3}. \tag{39}$$

It seems fair to compare the performance of both methods at the same rate. Therefore we choose

$$1 - \frac{1}{n_p} = \frac{m}{m+p}, \tag{40}$$

or

$$n_p = \left\lceil \frac{m+p}{p} \right\rceil, \tag{41}$$

where $\lceil x \rceil$ denotes the smallest integer $k \geq x$. Application of the outcomes of our analysis provides the results show in Table I. We may observe that the sum variance of Knuth codes is significantly larger than the sum variance of the polarity bit codes, for approximately the same redundancy.

VI. CONCLUSION

We have compared two methods for the generation of dc-balanced sequences. The two methods, Knuth's code and polarity bit code, have the virtue that they can be used without look-up tables. The spectral performance of the two methods has been evaluated with a parameter called the sum variance. Under the premise that the sum variance can serve as a quantity to judge the width of the spectral notch, we conclude that codes based on Knuth's algorithm offer less spectral suppression than polarity bit codes with the same redundancy.

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On the Shannon Capacity of DC- and Nyquist-Free Codes

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***Abstract*—The Shannon capacity of codes that are simultaneously dc- and Nyquist-free is tabulated as a function of the number of allowed running digital sum and alternating digital sum states. For this constraint, the set of possible rational capacities is shown to be {1/4,**

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